PHYS 301 Midterm

Oct. 21, 2024

You have 75 minutes to complete this midterm. Attempt all questions. Write your name and student number on this page. When necessary, make proper use of vector notation. Including this coversheet, which is unnumbered, there are a total of 11 pages. You may remove the last two sheets (also unnumbered) which are copies of the inside front and back covers of the Griffiths textbook.

If you require more space to write your solutions, use the backs of the pages.

Last Name: Sol'ns

First Name: ______________________

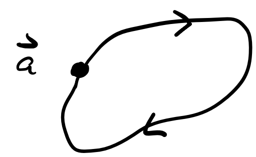
Student Number: ______________________

#1	#2	#3	#4	Total
3	**3**	**5**	**6**	**17**

Midterm (17 points)
Free Response: Write out complete answers to the following questions. Include diagrams where appropriate. Show your work since it allows us to award partial credit.

(3pts) **1.** Given that the electrostatic force is a *conservative* force for which the line integral:

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\boldsymbol{\ell} = W(\mathbf{b}) - W(\mathbf{a})$$



is path independent, show that $\nabla \times \mathbf{E} = 0$.

Choose $\vec{a} = \vec{b}$, s.t. we integrate around a closed path

and $\oint \vec{F} \cdot d\ell = 0$.

For electrostatic force $\vec{F} = q\vec{E}$ s.t.

$$\cancel{q} \oint \vec{E} \cdot d\vec{\ell} = 0$$

Next, apply Stoke's theorem $\oint \vec{v} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$

$$\therefore \oint \vec{E} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = 0$$

For a surface w/ nonzero area, require $\boxed{\vec{\nabla} \times \vec{E} = 0}$

3 pts

(3pts) **2.** (a) Evaluate the following integral: (**2 marks**)

$$\int_0^\infty Ae^{-x/\lambda}\sin\left(\frac{2\pi x}{a}\right)\delta\left(x-\frac{a}{4}\right)\,dx.$$

(b) Evaluate the following integral:

$$\int_{a/2}^\infty Ae^{-x/\lambda}\sin\left(\frac{2\pi x}{a}\right)\delta\left(x-\frac{a}{4}\right)\,dx.$$

Explain your reasoning. (**1 mark**)
Hint: Pay attention to the limits of integration.

(a) Use the fact that $\int_a^b f(x)\,\delta(x-x_0)\,dx = f(x_0)$ provided $a < x_0 < b$.

In our problem, $\delta\left(x-\frac{a}{4}\right)$ selects value of x that satisfies

$$x-\frac{a}{4}=0 \Rightarrow x=\frac{a}{4}$$

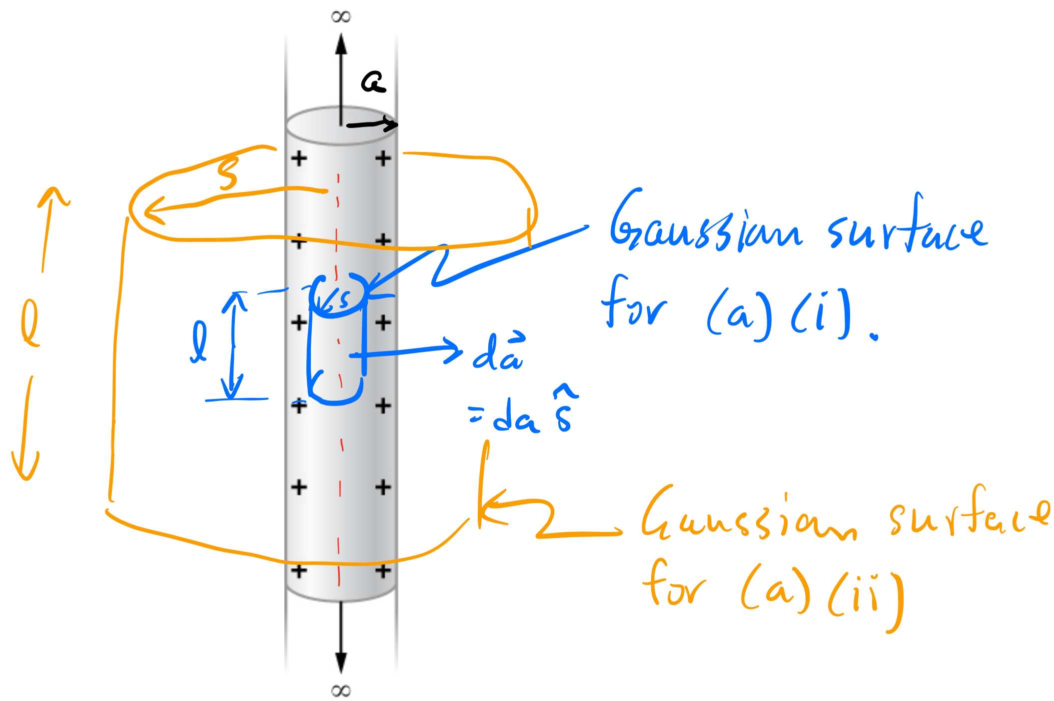
$$I=\int_0^\infty Ae^{-x/\lambda}\sin\left(\frac{2\pi x}{a}\right)\delta\left(x-\frac{a}{4}\right)dx = Ae^{-a/(4\lambda)}\underbrace{\sin\left(\frac{2\pi\cancel{a}}{4\cancel{a}}\right)}_{\sin\left(\frac{\pi}{2}\right)=1}$$

$$\therefore \boxed{I = Ae^{-a/(48)}}$$

(b) In this case, integration limits do <u>not</u> contain the pt. $x=a/4$. $\therefore \delta\left(x-\frac{\pi}{4}\right)=0$ everywhere on $a/2 < x < \infty$. $\therefore \boxed{I=0}$

3 pts

(5pts) **3.** (a) A long straight *conducting* pipe of radius a has a uniform charge per unit length λ. If s is the perpendicular distance from the pipe's central axis, find the electric field for points (i) inside ($s < a$) and (ii) outside ($s > a$) the pipe. Show your work. Solutions that only give the correct final answer will *not* be awarded full credit. (**3 marks**)



(b) Show that the electric fields calculated in (a) satisfy the boundary conditions for $E_\perp$. Assume that the pipe has thin walls and serves as the boundary separating the regions of space inside and outside of the pipe. (**2 marks**)

(a) Gauss's Law: $\oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{encl}}{\varepsilon_0}$

For both (i) & (ii), expect $\vec{E}$ to be radial.

$\therefore \vec{E} = E_s\,\hat{s}$. $\therefore$ No flux through the ends of the cylindrical Gaussian surfaces.

$$\oint \vec{E}\cdot d\vec{a} = \int_{\substack{\text{curved}\\\text{surface}}} E_s\,da \qquad (\text{since } d\vec{a} \parallel \vec{E})$$

5 pts

$$\therefore \oint \vec{E}\cdot d\vec{a} = E_s \int_{\substack{\text{curved}\\ \text{surface}}} da$$ (since E_s const. everywhere on curved Gaussian surface)

$$= E_s\, 2\pi s l$$

For (i) $Q_{encl} = 0$

$$\therefore E_s 2\pi s l = \frac{0}{\varepsilon_0} \Rightarrow \boxed{\vec{E} = 0 \quad s < a}$$

For (ii) $Q_{encl} = \lambda l$

$$\therefore E_s 2\pi s \cancel{l} = \frac{\lambda \cancel{l}}{\varepsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \quad s > a}$$

(b) b.c. $\underbrace{E^{\perp}_{outside}}_{\frac{\lambda}{2\pi\varepsilon_0 s}} - \underbrace{E^{\perp}_{inside}}_{0} = \frac{\sigma}{\varepsilon_0}$

$$\therefore \frac{\lambda}{2\pi\cancel{\varepsilon_0}s} = \frac{\sigma}{\cancel{\varepsilon_0}} \Rightarrow \lambda = 2\pi s\sigma$$

or $\lambda l = \underbrace{(2\pi s l)}_{A}\sigma = \sigma A = Q_{pipe}$

$$\boxed{\therefore \lambda = Q_{pipe}/l}$$

0 pts

(6pts) **4.** (a) Show that, by using separation of variables in Cartesian coordinates, Laplace's equation in two dimensions:

$$\nabla^2 V(x,y) = 0$$

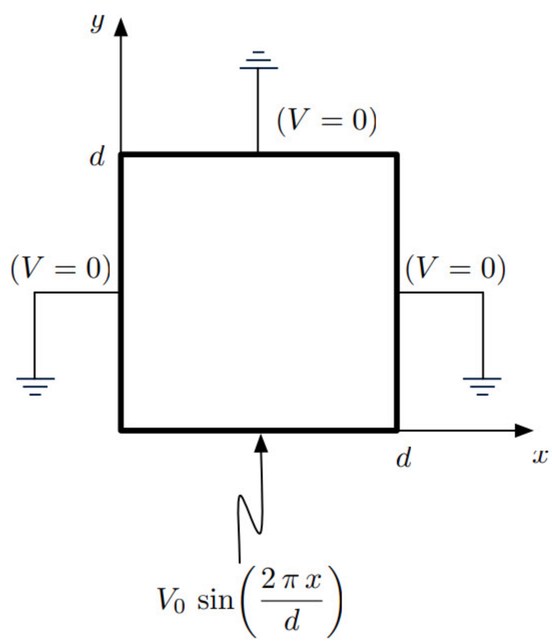
can be re-expressed as:

$$\frac{d^2X}{dx^2} = -k^2 X, \quad (1)$$

$$\frac{d^2Y}{dy^2} = k^2 Y, \quad (2)$$

where $V(x,y) = X(x)Y(y)$. (**2 marks**)

(b) Consider the two-dimensional square geometry below which contains three grounded wires ($V = 0$) of length d at (i) $x = 0$, (ii) $x = d$, and (iii) $y = d$. A fourth wire, at (iv) $y = 0$, is held fixed at a potential given by $V_0 \sin(2\pi x/d)$.



Given that the general solutions to Eqs. (1) and (2) in part (a) lead to:

$$V(x,y) = \left(Ce^{ky} + De^{-ky}\right)\left[A\sin(kx) + B\cos(kx)\right], \quad (3)$$

use the four boundary conditions [(i)...(iv)] to find the unknown constants A, B, C, D, and k in Eq. (3) and, thus, find the potential inside the square enclosed by the four wires. (**4 marks**)

(a) $\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) XY = Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$

divide by XY:

6 pts

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{C=-k^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{C=+k^2} = 0$$

require the two terms equal to the same const., but w/ opp. sign

$$\therefore \boxed{\frac{d^2X}{dx^2} = -k^2 X} \quad \boxed{\frac{d^2Y}{dy^2} = k^2 Y}$$

(b) $V(x,y) = \left(Ce^{ky} + De^{-ky}\right)\left(A\sin kx + B\cos kx\right)$

start w/ $V(0,y) = 0 = \left(Ce^{ky} + De^{-ky}\right) B$

$\therefore$ require $\boxed{B=0}$

$\therefore V(x,y) = \sin kx \left(C'e^{ky} + D'e^{-ky}\right)$ where $C' = AC$, $D' = AD$

Next, consider $V(d,y) = 0 = \sin(kd)\left(C'e^{ky} + D'e^{-ky}\right)$

$\therefore kd = n\pi \Rightarrow \boxed{k = \frac{n\pi}{d}}$

0 pts

Next, try $y=0$

$\therefore V(x,0) = (C'+D')\sin\left(\frac{n\pi x}{d}\right) = V_0 \sin\left(\frac{2\pi x}{d}\right)$

$$\boxed{\begin{array}{l}\therefore V_0 = C'+D' \\ n=2\end{array}} \quad 1 \quad \Rightarrow \quad \boxed{\therefore kd = \frac{2\pi}{d} d = 2\pi}$$

$\therefore V(x,y) = (C'e^{ky} + D'e^{-ky})\sin\left(\frac{2\pi x}{d}\right)$

Finally, try $y=d$

$V(x,d) = 0 = (C'e^{2\pi} + D'e^{-2\pi})\sin\left(\frac{2\pi x}{d}\right)$

$\therefore C'e^{2\pi} + D'e^{-2\pi} = 0$

$\therefore D' = -C'e^{4\pi}$

sub into 1

$V_0 = C'(1-e^{4\pi})$ $\therefore$ $\boxed{C' = \frac{V_0}{1-e^{4\pi}}}$

$$\boxed{D' = \frac{-V_0 e^{4\pi}}{1-e^{4\pi}}}$$

0 pts

$$\therefore V(x,y) = \frac{V_0}{1-e^{2kd}}\left(e^{ky} - e^{4\pi} e^{-ky}\right)\sin\left(\frac{2\pi x}{d}\right)$$

$$= \frac{V_0}{\underbrace{e^{2\pi} - e^{-2\pi}}_{2\sinh 2\pi}}\left(\underbrace{e^{2\pi} e^{-ky} - e^{-2\pi} e^{ky}}_{2\sinh\left[2\pi - \frac{2\pi y}{d}\right]}\right)\sin\left(\frac{2\pi x}{d}\right)$$

$$\boxed{\therefore V(x,y) = V_0\left(\frac{\sinh\left[2\pi\left(1-\frac{y}{d}\right)\right]}{\sinh 2\pi}\right)\sin\left(\frac{2\pi x}{d}\right)}$$

double check boundary conditions

$x=0$ $\quad V(0,y) \propto \sin(0) = 0$ ✓

$x=d$ $\quad V(d,y) \propto \sin(2\pi) = 0$ ✓

$y=d$ $\quad V(x,d) \propto \sinh(0) = 0$ ✓

$y=0$ $\quad V(x,0) = V_0\left(\cancelto{1}{\frac{\sinh 2\pi}{\sinh 2\pi}}\right)\sin\left(\frac{2\pi x}{d}\right)$ ✓

Full marks for finding k, B, C', & D'.
Everything on this page is just tidying up the $V(x,y)$ sol'n & double checking b.c.'s.

Griffiths front cover

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl:
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Griffiths back cover

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla\cdot\mathbf{D} = \rho_f \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad \mathbf{D} = \epsilon\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$

Momentum: $\mathbf{P} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c}q^2a^2$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85\times10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi\times10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00\times10^{8}\,\mathrm{m/s}$	(speed of light)
e	$= 1.60\times10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11\times10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$